SOLUTION OF THE EQUATIONS FOR A REGULAR BEAM EMITTED BY A CURVILINEAR SURFACE IN THE NONSTATIONARY CASE

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An analytical solution is given of the equations for a regular beam (§1) emitted by an arbitrary surface in the nonstationary case and the ρ- and T-limited states for nonzero initial velocity (§§2–4). It is assumed that the emitter is the coordinate surface $x^1 = 0$ in the orthogonal system x^i ($i = 1, 2, 3$), and the current density J, the electric field ε, and the magnetic field **H** are given functions $J(t, x^2, x^3)$, $\mathcal{E}(t, x^2, x^3)$, and $\mathbf{H}(x^1, x^2, x^3)$. The solution is given in the form of series in terms of $(x^1)^{\varkappa}$ with coefficients that are functions of t, x^2, and x^3. These coefficients are determined from recurrence relations ($\varkappa = 1/3, 1/2, 1$, depending on the emission conditions). Plane, cylindrical, and spherical diodes are considered in §5 in the case in which the high-frequency component of the current density J is not small in comparison with its constant components.

§1. BASIC EQUATIONS

A regular, mono-energetic, nonrelativistic beam of charged particles, all having the same specific charge η, is described by the following time-dependent system of differential equations which, in the tensor form and in an arbitrary curvilinear system of coordinates x^i ($i = 1, 2, 3$), is of the form

$$\frac{\partial v_i}{\partial t}+\frac{\partial}{\partial x^i}\Big(\frac{1}{2}g^{kl}v_kv_l-\varphi\Big)=0,\qquad H^l=\frac{1}{\sqrt{g}}e^{ikl}\frac{\partial v_i}{\partial x^k},$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ik}\rho v_k)=0,$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\Big(\sqrt{g}g^{ik}\frac{\partial\varphi}{\partial x^k}\Big)=\rho. \qquad (1.1)$$

These equations are written in the dimensionless variables introduced in [1], where all the main symbols are defined.

We shall suppose that the emitter coincides with the coordinate surface $x^1 = 0$ in the orthogonal system x^i ($i = 1, 2, 3$), and that its shape remains constant in time. Therefore, the metric in x^i is independent of t.

We shall show that the condition for the flow to be regular demands that the external magnetic field be constant. Integrating the first equation in Eqs. (1.1) for $i = 1$, we obtain

$$2\varphi=2\int\frac{\partial v_1}{\partial t}dx^1+V^2. \qquad (1.2)$$

When $i = 2$ we have

$$2\frac{\partial\varphi}{\partial x^2}=2\frac{\partial v_2}{\partial t}+\frac{\partial}{\partial x^2}(V^2). \qquad (1.3)$$

If we combine Eq. (1.2) and (1.3) and assume regular flow, we obtain

$$\int\frac{\partial}{\partial t}\Big(\frac{\partial v_1}{\partial x^2}-\frac{\partial v_2}{\partial x^1}\Big)dx^1=\int\sqrt{g}\frac{\partial H^3}{\partial t}dx^1=0.$$

Hence $\partial H^3/\partial t = 0$ and, similarly, $\partial H^1/\partial t = \partial H^2/\partial t = 0$. Therefore, a regular beam cannot be produced in a time-dependent external magnetic field.

§2. EMISSION IN THE ρ-LIMITED STATE

This is determined by the following conditions of the emitter: for $x^1 = 0$

$$V=0,\qquad \varphi=0,$$

$$\partial\varphi/\partial x^1=0,\qquad \rho v_{x^1}=J(t,x^2,x^3),$$

$$H_{x^1}=0,\qquad H_{x^2}=m(x^2,x^3),$$

$$H_{x^3}=n(x^2,x^3),\qquad m^2+n^2=h^2. \qquad (2.1)$$

We shall seek the solution of the problem defined by Eqs. (1.1) and (2.1) in the form

$$v_1=\sum_{k=2}^{\infty}U_k(x^1)^{1/3k},\qquad v_2=x^1\sum_{k=0}^{\infty}V_k(x^1)^{1/3k},$$

$$v_3=x^1\sum_{k=0}^{\infty}W_k(x^1)^{1/3k},$$

$$2\varphi=\sum_{k=4}^{\infty}\varphi_k(x^1)^{1/3k},\qquad 2\sqrt{g}\rho=\sum_{k=-2}^{\infty}\rho_k(x^1)^{1/3k}. \qquad (2.2)$$

where the subscripts and exponents k have the usual (nontensorial) significance, i.e., they label the terms in the series and give the powers of the various terms.

Substituting v_i and φ into the first equation in Eqs. (1.1) for $i = 1$, we obtain

$$\varphi_s=\frac{6}{s}(U_{s-3})_0'+\sum_{k=2}\Big[\Big(U_k^2+2\sum_{l=2}^{k-1}U_lU_{2k-l}\Big)A_{1/3(s-2k)}+\Big(2\sum_{l=2}^{k}U_lU_{2k-l+1}\Big)A_{1/3(s-2k-1)}\Big]+$$

$$+\sum_{k=0}\Big[\Big(V_k^2+2\sum_{l=0}^{k-1}V_lV_{2k-l}\Big)B_{1/3(s-2k-6)}+\Big(2\sum_{l=0}^{k}V_lV_{2k-l+1}\Big)B_{1/3(s-2k-7)}+$$

$$+\Big(W_k^2+2\sum_{l=0}^{k-1}W_lW_{2k-l}\Big)C_{1/3(s-2k-6)}+\Big(2\sum_{l=0}^{k}W_lW_{2k-l+1}\Big)C_{1/3(s-2k-7)}\Big]\qquad (s=4,5,\ldots). \qquad (2.3)$$

In these expressions $(U_s)_0' = \partial U_s/\partial t$. We recall that $U_1 = V_1 = W_1 = 0$ by definition, and A_k, B_k, C_k are the expansion coefficients for the elements of the metric tensor g^{ik} in terms of $(x^1)^k$. Therefore, for fixed s, the values of k must ensure that the indices are integers. Since $\partial H/\partial t = 0$, it follows that the other two equations ($i = 2, 3$) are satisfied identically.

The relation between V_k, W_k, and U_k is determined by the same formulas as in the stationary case [1]. The relation between ρ_k and φ_S is also the same except that φ_S is now given by Eqs. (2.3). If we use the equation for the conservation of current, we obtain the recurrence relations for the coefficients of the expansion in Eqs. (2.2):

$$(\rho_{p-3})_0' + \frac{p}{3}\sum_{t=0}^{p}\rho_{t-2}\sum_{l=0}A_l U_{p-t-3l+2} + \sum_{t=0}^{p-4}\Big[\Big(\rho_{t-2}\sum_{l=0}B_l V_{p-t-3l-4}\Big)_2' + \Big(\rho_{t-2}\sum_{l=0}C_l W_{p-t-3l-4}\Big)_3'\Big] = 0 \quad (p = 1, 2, \ldots). \tag{2.4}$$

Let us write out the first few terms in the expansion for the potential and v_1:

$$\begin{aligned}
2\varphi &= \left(\frac{9}{2}J\right)^{2/3} s^{4/3} - \frac{3}{5}\left(\frac{9}{2}J\right)^{1/3}\frac{J_0'}{J}s^{5/3} + \\
&+ \left(\frac{2}{5}\frac{J_0''}{J} - \frac{3}{8}\frac{J_0'^2}{J^2} + \frac{1}{10}h^2\right)s^2 + \\
&+ \left(\frac{9}{2}J\right)^{2/3}\left(\frac{1}{3}\frac{a_1}{a_0^{3/2}} + \frac{8}{15}T - \frac{1}{21}\frac{J_0'''}{J^2} + \right. \\
&\left. + \frac{6}{35}\frac{J_0'J_0''}{J^3} - \frac{1}{8}\frac{J_0'^3}{J^4} - \frac{1}{210}\frac{J_0'h^2}{J}\right)s^{7/3} + \\
&+ \left(\frac{9}{2}J\right)^{1/3}\Big[-\left(\frac{1}{4}\frac{a_1}{a_0^{3/2}} + \frac{23}{70}T\right)\frac{J_0'}{J} + \frac{3}{140}\frac{J_0^{IV}}{J^2} - \\
&- \frac{1}{8}\frac{J_0'J_0'''}{J^3} - \frac{9}{100}\frac{J_0''^2}{J^3} + \frac{9}{20}\frac{J_0'^2J_0''}{J^4} - \frac{33}{128}\frac{J_0'^4}{J^5} + \\
&+ \left(\frac{3}{350}\frac{J_0''}{J^2} - \frac{1}{80}\frac{J_0'^2}{J^3}\right)h^2 + \\
&+ \frac{9}{1400}\frac{h^4}{J} + \frac{1}{35}\left(n\frac{J_P'}{J} - m\frac{J_Q'}{J}\right) - \\
&- \frac{1}{14}(nk_2 - m\delta_2) + \frac{1}{14}(n_P' - m_Q')\Big]s^{8/3} + \\
&+ \Big[\frac{a_1}{a_0^{3/2}}\left(\frac{1}{5}\frac{J_0''}{J} - \frac{3}{16}\frac{J_0'^2}{J^2} + \frac{1}{20}h^2\right) + \\
&+ T\left(\frac{153}{700}\frac{J_0''}{J} - \frac{57}{280}\frac{J_0'^2}{J^2} + \frac{27}{700}h^2\right) + \\
&+ \frac{1}{28}(\varkappa_1 n^2 + \varkappa_2 m^2) + \frac{1}{56}(h^2)_S' + \\
&+ \frac{3}{56}\left(-n\frac{J_{0P}''}{J} + m\frac{J_{0Q}''}{J} + n\frac{J_0'J_P'}{J^2} - m\frac{J_0'J_Q'}{J^2}\right) + \\
&+ \frac{13}{280}(-n_P' + m_Q')\frac{J_0'}{J} + \frac{3}{28}(nk_1 - m\delta_1)\frac{J_0'}{J} + \\
&+ \frac{13}{280}(nk_2 - m\delta_2)\frac{J_0'}{J} + \frac{441}{19600}\frac{J_0'^2}{J^2}h^4 + \\
&+ \left(-\frac{1}{120}\frac{J_0'''}{J^2} + \frac{22}{525}\frac{J_0'J_0''}{J^2} - \frac{13}{280}\frac{J_0'^3}{J^4}\right)h^2 - \\
&- \frac{1}{120}\frac{J_0^{V}}{J^2} + \frac{1}{14}\frac{J_0'J_0^{IV}}{J^3} + \frac{2}{15}\frac{J_0''J_0'''}{J^3} - \frac{3}{8}\frac{J_0'^2J_0'''}{J^4} - \\
&- \frac{27}{50}\frac{J_0'J_0''^2}{J^4} + \frac{27}{20}\frac{J_0'^3J_0''}{J^5} - \frac{81}{128}\frac{J_0'^5}{J^6}\Big]s^3 + \ldots
\end{aligned}$$

$$\begin{aligned}
a_0^{-1/2}v_1 &= \left(\frac{9}{2}J\right)^{1/3}s^{2/3} - \frac{1}{2}\frac{J_0'}{J}s + \\
&+ \left(\frac{2}{9J}\right)^{1/3}\left(\frac{9}{20}\frac{J_0''}{J} - \frac{9}{16}\frac{J_0'^2}{J^2} - \frac{9}{20}h^2\right)s^{4/3} + \\
&+ \left(\frac{9}{2}J\right)^{1/3}\left(\frac{2}{3}\frac{a_1}{a_0^{3/2}} + \frac{4}{15}T - \frac{1}{15}\frac{J_0'''}{J^2} + \right. \\
&\left. + \frac{3}{10}\frac{J_0'^2}{J^3} - \frac{1}{4}\frac{J_0'^3}{J^4} - \frac{1}{15}\frac{J_0'}{J}h^2\right)s^{5/3} + \\
&+ \Big[-\left(\frac{3}{8}\frac{a_1}{a_0^{3/2}} + \frac{9}{140}T\right)\frac{J_0'}{J} - \frac{27}{1400}\frac{h^4}{J} + \\
&+ \frac{3}{14}\left(n\frac{J_P'}{J} - m\frac{J_Q'}{J}\right) - \frac{1}{28}(nk_2 - m\delta_2) + \\
&+ \frac{1}{28}(n_P' - m_Q') + \frac{1}{28}\frac{J_0^{IV}}{J^2} - \frac{1}{4}\frac{J_0'J_0'''}{J^3} - \\
&- \frac{9}{50}\frac{J_0''^2}{J^3} + \frac{81}{80}\frac{J_0'^2J_0''}{J^4} - \frac{81}{128}\frac{J_0'^4}{J^5} + \\
&+ \left(\frac{13}{175}\frac{J_0''}{J^2} - \frac{11}{80}\frac{J_0'^2}{J^3}\right)h^2\Big]s^2 + \left(\frac{2}{9J}\right)^{1/3}\Big\{\frac{a_1}{a_0^{3/2}}\left(\frac{3}{8}\frac{J_0''}{J} - \right. \\
&\left. - \frac{15}{32}\frac{J_0'^2}{J^2} - \frac{3}{8}h^2\right) + T\left(\frac{3}{280}\frac{J_0''}{J} - \frac{3}{560}\frac{J_0'^2}{J^2} + \right. \\
&\left. + \frac{39}{280}h^2\right) - \frac{27}{56}\left[\varkappa_1 n^2 + \varkappa_2 m^2 + \frac{1}{2}(h^2)_S'\right] - \\
&- \frac{39}{112}\left(n\frac{J_{0P}''}{J} - m\frac{J_{0Q}''}{J}\right) + \\
&+ \frac{51}{112}\left(n\frac{J_0'J_P'}{J^2} - m\frac{J_0'J_Q'}{J^2}\right) + \\
&+ \frac{3}{560}(-n_P' + m_Q')\frac{J_0'}{J} + \frac{3}{56}(nk_1 - m\delta_1)\frac{J_0'}{J} + \\
&+ \frac{3}{560}(nk_2 - m\delta_2)\frac{J_0'}{J} - \frac{39}{1120}\frac{J_0'}{J^2}h^4 + \\
&+ \left(-\frac{33}{560}\frac{J_0'''}{J^2} + \frac{51}{140}\frac{J_0'J_0''}{J^3} - \frac{9}{56}\frac{J_0'^3}{J^4}\right)h^2 - \\
&- \frac{3}{560}\frac{J_0^{V}}{J^2} + \frac{9}{56}\frac{J_0'J_0^{IV}}{J^3} + \frac{3}{10}\frac{J_0''J_0'''}{J^3} - \\
&- \frac{15}{16}\frac{J_0'^2J_0'''}{J^4} - \frac{27}{20}\frac{J_0'J_0''^2}{J^4} + \\
&+ \frac{117}{32}\frac{J_0'^3J_0''}{J^5} - \frac{117}{64}\frac{J_0'^5}{J^6}\Big\}s^{7/3} + \cdots \quad (s = a_0^{1/2}x^1).
\end{aligned} \tag{2.5}$$

In these expressions, a_k are the expansion coefficients for g_{11} in powers of $(x^1)^k$, S, P, and Q are the arc lengths measured along the curvilinear axes x^1, x^2, x^3, $\varkappa_1$ and $\varkappa_2$ are the principal curvatures of the emitting surface, $T = \varkappa_1 + \varkappa_2$ is its total curvature, and k_1 and k_2, δ_1 and δ_2 are the principal curvatures of the surfaces $x^2 = \text{const}$, $x^3 = \text{const}$, calculated for $x^1 = 0$

It is clear then that, when $\partial/\partial t \neq 0$, the coefficient of $s^{5/3}$ in the expansion for the potential is not zero. The fourth term includes a component representing the fact that the process is nonstationary. This component is proportional to the square of the magnetic field. The coefficient of $s^{8/3}$ describes the effect of the emitter geometry (through T) and the fact that the current density is not constant. Finally, the last term in Eqs. (2.5) reflects the interaction of the magnetic field and its inhomogeneity with the inhomogeneity of J, the dependence of J on t, and the geometric factors (through nk_1, nk_2, etc.).

§3. EMISSION IN THE T-LIMITED STATE

This differs from the emission limited by the space charge by the fact that the electric field at the emitter $x^1 = 0$ is not zero, and is a function of time:

$$V = 0, \quad \varphi = 0, \quad \sqrt{g^{11}}\partial\varphi/\partial x^1 = \varepsilon(t, x^2, x^3),$$
$$\rho v_{x^1} = J(t, x^2, x^3), \quad H_{x^1} = 0,$$
$$H_{x^2} = m(x^2, x^3), \quad H_{x^3} = n(x^2, x^3). \tag{3.1}$$

As in the stationary case, we shall develop expansions in half-integral powers of x^1, which satisfy Eqs. (1.1) and (3.1):

$$v_1 = \sum_{k=1}^{\infty} U_k (x^1)^{1/2 k},$$

$$v_2 = x^1 \sum_{k=0}^{\infty} V_k (x^1)^{1/2 k}, \qquad v_3 = x^1 \sum_{k=0}^{\infty} W_k (x^1)^{1/2 k}$$

$$2\varphi = \sum_{k=2}^{\infty} \varphi_k (x^1)^{1/2 k}, \qquad 2\sqrt{g}\,\rho = \sum_{k=-1}^{\infty} \rho_k (x^1)^{1/2 k}. \quad (3.2)$$

The equations of motion for a regular beam lead to the following expression:

$$\varphi_s = \frac{4}{s}(U_{s-2})_0' + \sum_{k=1}\Big[\Big(U_k^2 + 2\sum_{l=1}^{k-1} U_l U_{2k-l}\Big) A_{1/2 s-k} + \Big(2\sum_{l=1}^{k} U_l U_{2k-l+1}\Big) A_{1/2(s-1)-k}\Big] +$$
$$+ \sum_{k=0}\Big[\Big(V_k^2 + 2\sum_{l=0}^{k-1} V_l V_{2k-l}\Big) B_{1/2(s-4)-k} + \Big(2\sum_{l=0}^{k} V_l V_{2k-l+1}\Big) B_{1/2(s-5)-k} +$$
$$+ \Big(W_k^2 + 2\sum_{l=0}^{k-1} W_l W_{2k-l}\Big) C_{1/2(s-4)-k} + \Big(2\sum_{l=0}^{k} W_l W_{2k-l+1}\Big) C_{1/2(s-5)-k}\Big] \qquad (s = 2, 3, \ldots). \quad (3.3)$$

The formulas for V_k, W_k, and ρ_k are given in [1]. The recurrence relations for the coefficients of the expansions given by (3.2) are of the form

$$(\rho_{p-2})_0' + \sum_{t=1}\Big\{\frac{p}{2}\rho_{t-2}\sum_{l=1} U_l A_{1/2(p-t-l)+1} + \Big[\rho_{t-2}\sum_{l=0} V_l B_{1/2(p-t-l)-1}\Big]_2' + \Big[\rho_{t-2}\sum_{l=0} W_l C_{1/2(p-t-l)-1}\Big]_3'\Big\} = 0 \qquad (p = 1, 2, \ldots) \quad (3.4)$$

For the potential φ and v_1 we have

$$2\varphi = 2\varepsilon s + \frac{4\sqrt{2}}{3}\frac{J}{\sqrt{\varepsilon}} s^{3/2} + \Big[\Big(\frac{1}{2}\frac{a_1}{a_0^{3/2}} + T\Big)\varepsilon - \frac{1}{3}\frac{J^2}{\varepsilon^2} - \frac{J_0'}{\varepsilon} + \frac{2}{3}\frac{J\varepsilon_0'}{\varepsilon^2}\Big] s^2 +$$
$$+ \frac{4\sqrt{2}}{15}\Big[\Big(\frac{15}{8}\frac{a_1}{a_0^{3/2}} + \frac{11}{4}T\Big)\frac{J}{\sqrt{\varepsilon}} + \frac{1}{\sqrt{\varepsilon}}\Big(\frac{5}{12}\frac{J^3}{\varepsilon^3} + \frac{1}{4}\frac{Jh^2}{\varepsilon} + \frac{J_0''}{\varepsilon} - \frac{3}{4}\frac{J\varepsilon_0''}{\varepsilon^2} - 2\frac{J_0'\varepsilon_0'}{\varepsilon^2} + \frac{7}{4}\frac{JJ_0'}{\varepsilon^2} - \frac{5}{3}\frac{J^2\varepsilon_0'}{\varepsilon^3} + \frac{5}{3}\frac{J\varepsilon_0'^2}{\varepsilon^3}\Big)\Big] s^{5/2} + \Big\{\Big(\frac{1}{3}\frac{a_2}{a_0^2} - \frac{1}{12}\frac{a_1^2}{a_0^3}\Big)\varepsilon +$$
$$+ \frac{a_1}{a_0^{3/2}}\Big(-\frac{1}{6}\frac{J^2}{\varepsilon^2} + \frac{1}{2}\varepsilon T - \frac{1}{2}\frac{J_0'}{\varepsilon} + \frac{1}{3}\frac{J\varepsilon_0'}{\varepsilon^2}\Big) + \frac{1}{3}\varepsilon T^2 + \frac{1}{3}\varepsilon T_S' + T\Big(-\frac{1}{5}\frac{J^2}{\varepsilon^2} - \frac{5}{9}\frac{J_0'}{\varepsilon} + \frac{16}{45}\frac{J\varepsilon_0'}{\varepsilon^2}\Big) + \frac{1}{9\varepsilon}\Big[(-nk_2 + m\delta_2)J + (Jn)_P' - (Jm)_Q' +$$
$$+ (-n\varepsilon_P' + m\varepsilon_Q')\frac{J}{\varepsilon}\Big] + \frac{1}{3}[(3k_1 + k_2)\varepsilon_P' + (3\delta_1 + \delta_2)\varepsilon_Q' - 2(k_1^2 + \delta_1^2)\varepsilon - (k_1k_2 + \delta_1\delta_2)\varepsilon - \varepsilon_P'' - \varepsilon_Q'' + (k_{1P}' + \delta_{1Q}')\varepsilon] - \frac{8}{81}\frac{J^4}{\varepsilon^5} - \frac{4}{45}\frac{J^2h^2}{\varepsilon^3} -$$
$$- \frac{1}{9}\frac{J_0'''}{\varepsilon^2} - \frac{16}{45}\frac{JJ_0''}{\varepsilon^3} + \frac{1}{3}\frac{J_0''\varepsilon_0'}{\varepsilon^3} - \frac{1}{3}\frac{J_0'^2}{\varepsilon^3} - \frac{5}{9}\frac{J^2J_0'}{\varepsilon^4} - \frac{1}{9}\frac{J_0'h^2}{\varepsilon^2} + \frac{4}{45}\frac{J\varepsilon_0'''}{\varepsilon^3} - \frac{2}{3}\frac{J\varepsilon_0'\varepsilon_0''}{\varepsilon^4} +$$
$$+ \frac{1}{3}\frac{J_0'\varepsilon_0''}{\varepsilon^3} + \frac{1}{3}\frac{J^2\varepsilon_0''}{\varepsilon^4} + \frac{64}{81}\frac{J\varepsilon_0'^3}{\varepsilon^5} - \frac{8}{9}\frac{J_0'\varepsilon_0'^2}{\varepsilon^4} - \frac{32}{27}\frac{J^2\varepsilon_0'^2}{\varepsilon^5} + \frac{14}{9}\frac{JJ_0'\varepsilon_0'}{\varepsilon^4} + \frac{16}{27}\frac{J^3\varepsilon_0'}{\varepsilon^5} + \frac{2}{15}\frac{J\varepsilon_0'h^2}{\varepsilon^3}\Big\} s^3 + \ldots$$

$$a_0^{-1/2} v_1 = \sqrt{2\varepsilon}\, s^{1/2} + \frac{2}{3\varepsilon}\Big(J - \frac{1}{2}\varepsilon_0'\Big) s + \frac{1}{\sqrt{2\varepsilon}}\Big[\Big(\frac{5}{4}\frac{a_1}{a_0^{3/2}} + \frac{1}{2}T\Big)\varepsilon - \frac{7}{18}\frac{J^2}{\varepsilon^2} - \frac{1}{2}h^2 + \frac{1}{6}\frac{\varepsilon_0''}{\varepsilon} - \frac{2}{9}\frac{\varepsilon_0'^2}{\varepsilon^2} + \frac{8}{9}\frac{J\varepsilon_0'}{\varepsilon^2} - \frac{5}{6}\frac{J_0'}{\varepsilon}\Big] s^{3/2} +$$
$$+ \Big[\Big(\frac{1}{2}\frac{a_1}{a_0^{3/2}} + \frac{1}{5}T\Big)\frac{J}{\varepsilon} + \frac{5}{27}\frac{J^3}{\varepsilon^4} + \frac{1}{5}\frac{Jh^2}{\varepsilon^2} + \Big(-\frac{1}{4}\frac{a_1}{a_0^{3/2}} + \frac{1}{30}T\Big)\frac{\varepsilon_0'}{\varepsilon} + \frac{3}{10}\frac{J_0''}{\varepsilon^2} - \frac{5}{6}\frac{J_0'\varepsilon_0'}{\varepsilon^3} +$$
$$+ \frac{2}{3}\frac{JJ_0'}{\varepsilon^3} - \frac{1}{30}\frac{\varepsilon_0'''}{\varepsilon^2} + \frac{1}{6}\frac{\varepsilon_0'\varepsilon_0''}{\varepsilon^3} - \frac{1}{3}\frac{J\varepsilon_0''}{\varepsilon^3} - \frac{4}{27}\frac{\varepsilon_0'^3}{\varepsilon^4} + \frac{8}{9}\frac{J\varepsilon_0'^2}{\varepsilon^4} - \frac{7}{9}\frac{J^2\varepsilon_0'}{\varepsilon^4} + \frac{2}{15}\frac{h^2\varepsilon_0'}{\varepsilon^2} +$$
$$+ \frac{1}{3}\Big(n\frac{\varepsilon_P'}{\varepsilon} - m\frac{\varepsilon_Q'}{\varepsilon}\Big) - \frac{1}{2}(nk_1 - m\delta_1)\Big] s^2 + \ldots \quad (3.5)$$

Comparison of the expressions for the potential in Eqs. (2.5) and (3.5) shows that, while in the case of emission in the ρ state and with $\partial/\partial t \neq 0$, the effect of the magnetic field predominates over the geometric effects ($J_0'h^2$ is present in the coefficient of $s^{7/3}$ and $J_0'T$ is present in the coefficient of $s^{8/3}$), in the case of emission limited by temperature, the terms $J_0'T$, $\varepsilon_0'T$ and $J_0'h^2$, $\varepsilon_0'h^2$ appear simultaneously (in the coefficient of s^3). Moreover, in the space-charge-limited case there is an earlier appearance of terms representing the cross effects of the magnetic field, nonstationarity, geometry, and inhomogeneity of the emission current ($nJ_0'k_1$, $nJ_0''P$, etc. in the coefficients of s^3).

§4. CASE OF NONZERO EMISSION VELOCITY.

This is described by the following conditions for $x^1 = 0$:

$$v_{x^1} = u(t), \quad \varphi = 0,$$

$$\sqrt{g^{11}}\,\partial\varphi/\partial x^1 = \varepsilon(t, x^2, x^3), \quad \rho v_{x^1} = J(t, x^2, x^3)$$

$$H_{x^1} = 0, \quad H_{x^2} = m(x^2, x^3), \quad H_{x^3} = n(x^2, x^3). \quad (4.1)$$

The solution of the problem defined by Eqs. (1.1) and (4.1) will be sought in the form

$$v_1 = \sum_{k=0}^{\infty} U_k (x^1)^k, \qquad v_2 = x^1 \sum_{k=0}^{\infty} V_k (x^1)^k,$$

$$v_3 = x^1 \sum_{k=0}^{\infty} W_k (x^1)^k,$$

$$2\varphi = \sum_{k=0}^{\infty} \varphi_k (x^1)^k, \qquad 2\sqrt{g}\,\rho = \sum_{k=0}^{\infty} \rho_k (x^1)^k. \quad (4.2)$$

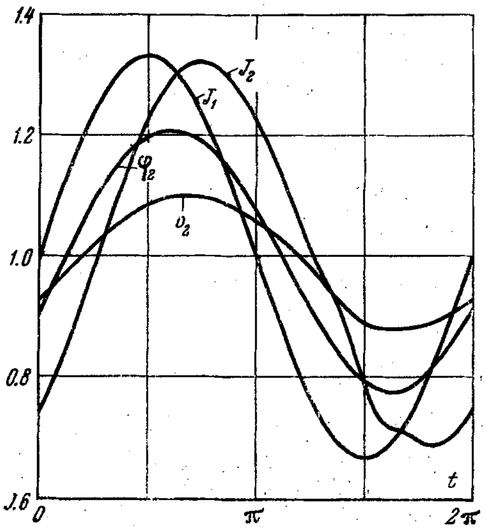

Fig. 1

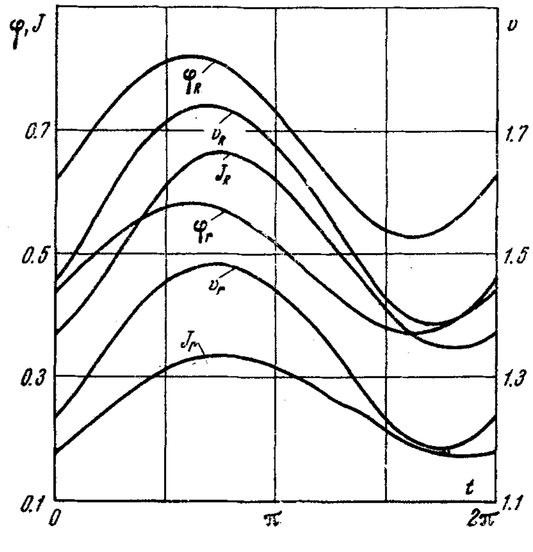

Fig. 2

The expansion coefficients for the potential are given by

$$\begin{aligned}\varphi_s = \frac{2}{s}(U_{s-1})_0' + \sum_{k=0}\Big[\Big(U_k^2 &+ \\ + 2\sum_{l=1}^{k} U_{l-1}U_{2k-l+1}\Big)A_{s-2k} &+ \\ + \Big(2\sum_{l=0}^{k} U_l U_{2k-l+1}\Big)A_{s-2k-1} &+ \\ + \Big(V_k^2 + 2\sum_{l=1}^{k} V_{l-1}V_{2k-l+1}\Big)B_{s-2k-2} &+ \\ + \Big(2\sum_{l=0}^{k} V_l V_{2k-l+1}\Big)B_{s-2k-3} + \Big(W_k^2 &+ \\ + 2\sum_{l=1}^{k} W_{l-1}W_{2k-l+1}\Big)C_{s-2k-2} &+ \\ + \Big(2\sum_{l=0}^{k} W_l W_{2k-l+1}\Big)C_{s-2k-3}\Big] \qquad (s = 1, 2, \ldots). \qquad (4.3)\end{aligned}$$

The formulas for V_k, W_k, and ρ_k are the same as in [1]. The recurrence relations can be written as follows:

$$\begin{aligned}(\rho_{p-1})_0' + p\sum_{l=0}^{p}\rho_l\sum_{l=0}^{p-l} A_l U_{p-l-l} &+ \\ + \sum_{l=0}^{p-2}\Big[\Big(\rho_l\sum_{l=0}^{p-l-2} B_l V_{p-l-l-2}\Big)_2' &+ \\ + \Big(\rho_l\sum_{l=0}^{p-l-2} C_l W_{p-l-l-2}\Big)_3'\Big] = 0 \qquad (p = 1, 2, \ldots). \qquad (4.4)\end{aligned}$$

For the potential φ we have

$$2\varphi = u^2 + 2\varepsilon s + \Big(\frac{1}{2}\frac{a_1}{a_0^{3/2}}\varepsilon + \varepsilon T + \frac{J}{u}\Big)s^2 + \ldots .$$

The corrections to φ_3 and φ_4 due to nonzero derivatives with respect to time for u, J, and ε are given by

$$\Delta_3 = \Big(\frac{2}{3}\frac{Ju'}{u^3} - \frac{1}{3}\frac{J_0'}{u^2}\Big)a_0^{3/2},$$

$$\begin{aligned}\Delta_4 = \Big[\frac{a_1}{a_0^{3/2}}\Big(\frac{1}{2}\frac{Ju'}{u^3} - \frac{1}{4}\frac{J_0'}{u^2}\Big) - \frac{1}{4}\frac{Ju''}{u^4} + \frac{Ju'^2}{u^5} &- \\ - \frac{1}{2}\frac{J_0'u'}{u^4} + \frac{1}{4}\frac{Ju'}{u^3}T - \frac{J\varepsilon u'}{u^5} + \frac{1}{6}\frac{J\varepsilon_0'}{u^4} &+ \\ + \frac{1}{2}\frac{J_0''}{u^3} - \frac{1}{4}\frac{J_0'}{u}T + \frac{1}{4}\frac{J_0'\varepsilon}{u^4}\Big]a_0^2. \qquad (4.5)\end{aligned}$$

It is interesting to compare the results obtained using the above expansions with existing exact analytic solutions of the equations for a beam. The number of of such solutions is, unfortunately, very small. In the stationary case the variables can be separated [2, 3]. Therefore, for example, in the case of the solutions given by Meltzer [2], the problem reduces to finding the Taylor expansion for the function $F = (\sin 3\psi/2)^{4/3}$. When $\psi < \pi/3$, so that F increases monotonically, it is possible to achieve high accuracy with a relatively small number of terms. When $\psi \geq \pi/3$, the situation is different. To obtain the solution in this region it is best to go over to an expansion in integral powers of x^1, beginning with a certain $\psi_0 < \pi/3$, and then match it to the solution describing the emission under space-charge-limited conditions for $0 \leq \psi \leq \psi_0$.

An analytic solution containing an arbitrary function of time $f(t)$, and describing certain nonstationary processes in the planar diode was given in [4]. In the case of the Child-Langmuir three-halves law, we have for the dimensionless potential

$$\varphi = (x + f)^{4/3} - 2f'' x - f^{4/3}$$

It is readily verified that the convergence radius for the φ series is f, while the solution must be obtained for $0 \leq x \leq 1$. If we allow an error of the order of 0.1% for $f = 2 + \sin t$ and $x = 1$, we find that the required accuracy for $t = 0$, $t = \pi/2$ and $t = 3\pi/2$ is achieved with five, four, and nine series terms, respectively.

§5. THEORY OF PLANAR, CYLINDRICAL, AND SPHERICAL DIODES.

These simplest configurations are of particular interest because, under homogeneous conditions on the emitter, all the equipotential surfaces φ = const are time-independent. Since we shall be interested in high-frequency processes in these devices during emission under space-charge-limited conditions, and zero external magnetic fields, let us write down a few further terms in the expansions for φ and v_1. The formulas which are given below refer to the case $H = 0$ and $J = J(t)$, and do not involve any additional assumptions:

$$\begin{aligned}\varphi_{10} = \Big(\frac{2}{9J}\Big)^{1/3}\Big[\frac{a_1}{a_0^{3/2}}\Big(-\frac{1}{8}\frac{J_0'''}{J} + \frac{9}{20}\frac{J_0'J_0''}{J^2} &- \\ - \frac{21}{64}\frac{J_0'^3}{J^3} + \frac{7}{5}JT\Big) - \frac{3}{16}\frac{a_1^2}{a_0^3}J + \frac{a_2}{a_0^2}J &+ \\ + T\Big(-\frac{4}{35}\frac{J_0'''}{J} + \frac{163}{400}\frac{J_0'J_0''}{J^2} - \frac{1653}{5600}\frac{J_0'^3}{J^3}\Big) &+ \\ + \frac{157}{200}JT^2 + \frac{7}{8}JT_S' - \frac{15}{8}J(k_1^2 + \delta_1^2) - \frac{7}{8}J(k_1k_2 + \delta_1\delta_2) &+ \\ + \frac{7}{8}J(k_{1P}' + \delta_{1Q}') + \frac{1}{350}\frac{J_0^{VI}}{J^2} - \frac{27}{800}\frac{J_0'J_0^{V}}{J^3} &- \\ - \frac{27}{350}\frac{J_0''J_0^{IV}}{J^3} + \frac{27}{112}\frac{J_0'^2J_0^{IV}}{J^4} - \frac{1}{20}\frac{J_0'''^2}{J^3} &+ \\ + \frac{9}{10}\frac{J_0'J_0''J_0'''}{J^4} - \frac{39}{32}\frac{J_0'^3J_0'''}{J^5} + \frac{27}{125}\frac{J_0''^3}{J^4} &- \\ - \frac{1053}{400}\frac{J_0'^2J_0''^2}{J^5} + \frac{351}{80}\frac{J_0'^4J_0''}{J^6} - \frac{2223}{1280}\frac{J_0'^6}{J^7}\Big]a_0^{6/3},\end{aligned}$$

$$\begin{aligned}\varphi_{11} = \Big\{\Big(\frac{9}{2}J\Big)^{1/3}\Big[\frac{a_1}{a_0^{3/2}}\Big(-\frac{23}{105}\frac{J_0'}{J}T &+ \\ + \frac{1}{70}\frac{J_0^{IV}}{J^2} - \frac{1}{12}\frac{J_0'J_0'''}{J^3} - \frac{3}{50}\frac{J_0''^2}{J^3} &+ \\ + \frac{3}{10}\frac{J_0'^2J_0''}{J^4} - \frac{11}{64}\frac{J_0'^4}{J^5}\Big) + \frac{1}{48}\frac{a_1^2}{a_0^3}\frac{J_0'}{J} &- \\ - \frac{1}{6}\frac{a_2}{a_0^2}\frac{J_0'}{J} + T\Big(\frac{23}{2100}\frac{J_0^{IV}}{J^2} - \frac{1459}{23100}\frac{J_0'J_0'''}{J^3} &- \\ - \frac{873}{19250}\frac{J_0''^2}{J^3} + \frac{41611}{184800}\frac{J_0'^2J_0''}{J^4} - \frac{41}{320}\frac{J_0'^4}{J^5}\Big) &+ \\ + \frac{J_0'}{J}\Big(-\frac{967}{9240}T^2 - \frac{57}{440}T_S' + \frac{37}{120}[k_1^2 + \delta_1^2] &+ \\ + \frac{57}{440}[k_1k_2 + \delta_1\delta_2] - \frac{57}{440}[k_{1P}' + \delta_{1Q}']\Big)\Big] &+ \\ + \Big(\frac{2}{9J}\Big)^{2/3}\Big[-\frac{27}{30800}\frac{J_0^{VII}}{J^2} + \frac{3}{220}\frac{J_0'J_0^{VI}}{J^3} + \frac{81}{2200}\frac{J_0''J_0^{V}}{J^3} &- \\ - \frac{81}{640}\frac{J_0'^2J_0^{V}}{J^4} + \frac{9}{154}\frac{J_0'''J_0^{IV}}{J^3} - \frac{81}{140}\frac{J_0'J_0''J_0^{IV}}{J^4} &+ \\ + \frac{27}{32}\frac{J_0'^3J_0^{IV}}{J^5} - \frac{3}{8}\frac{J_0'J_0'''^2}{J^4} &- \\ - \frac{27}{50}\frac{J_0''^2J_0'''}{J^4} + \frac{189}{40}\frac{J_0'^2J_0''J_0'''}{J^5} &-\end{aligned}$$

$$- \frac{1071}{256}\frac{J_0'^4 J_0'''}{J^6} + \frac{567}{256}\frac{J_0' J_0''^3}{J^5} - \frac{9639}{800}\frac{J_0'^3 J_0''^2}{J^6} +$$

$$+ \frac{9639}{640}\frac{J_0'^5 J_0''}{J^7} - \frac{10557}{2048}\frac{J_0'^7}{J^8}\Big]\Big\} a_0^{11/2},$$

$$\varphi_{12} = \Big\{\frac{a_1}{a_0^{3/2}}\Big[T\Big(\frac{459}{2800}\frac{J_0''}{J} - \frac{171}{1120}\frac{J_0'^2}{J^2}\Big) - \frac{1}{160}\frac{J_0^{\mathrm{V}}}{J^2} +$$

$$+ \frac{3}{56}\frac{J_0' J_0^{\mathrm{IV}}}{J^3} + \frac{1}{10}\frac{J_0'' J_0'''}{J^3} - \frac{9}{32}\frac{J_0'^2 J_0'''}{J^4} -$$

$$- \frac{81}{200}\frac{J_0'^3 J_0''}{J^5} - \frac{243}{512}\frac{J_0'^5}{J^6}\Big] + \frac{a_1^2}{a_0^3}\Big(-\frac{1}{120}\frac{J_0''}{J} +$$

$$+ \frac{1}{128}\frac{J_0'^2}{J^2}\Big) + \frac{a_2}{a_0^2}\Big(\frac{2}{15}\frac{J_0''}{J} - \frac{1}{8}\frac{J_0'^2}{J^2}\Big) +$$

$$+ T\Big(-\frac{53}{13200}\frac{J_0^{\mathrm{V}}}{J^2} + \frac{1831}{53900}\frac{J_0' J_0^{\mathrm{IV}}}{J^3} +$$

$$+ \frac{389}{6160}\frac{J_0'' J_0'''}{J^3} - \frac{987}{5600}\frac{J_0'^2 J_0'''}{J^4} - \frac{1063}{4200}\frac{J_0' J_0''^2}{J^4} +$$

$$+ \frac{14069}{22400}\frac{J_0'^3 J_0''}{J^5} - \frac{26191}{8960}\frac{J_0'^5}{J^6}\Big) + \frac{J_0''}{J}\Big[\frac{281}{4200}T^2 + \frac{51}{550}T_S' -$$

$$- \frac{1021}{13200}(k_1^2 + \delta_1^2) - \frac{69}{2200}(k_1 k_2 + \delta_1\delta_2) +$$

$$+ \frac{69}{2200}(k_{1P}' + \delta_{1Q}')\Big] + \frac{J_0'^2}{J^2}\Big[-\frac{10691}{172480}T^2 - \frac{61}{704}T_S' +$$

$$+ \frac{433}{3520}(k_1^2 + \delta_1^2) + \frac{161}{3520}(k_1 k_2 + \delta_1\delta_2) +$$

$$+ \frac{161}{3520}(k_{1P}' + \delta_{1Q}')\Big] + \frac{1}{18480}\frac{J_0^{\mathrm{VIII}}}{J^3} -$$

$$- \frac{3}{1400}\frac{J_0' J_0^{\mathrm{VII}}}{J^4} - \frac{1}{300}\frac{J_0'' J_0^{\mathrm{VI}}}{J^4} +$$

$$+ \frac{1}{80}\frac{J_0'^2 J_0^{\mathrm{VI}}}{J^5} - \frac{1}{160}\frac{J_0''' J_0^{\mathrm{V}}}{J^4} + \frac{189}{2800}\frac{J_0' J_0'' J_0^{\mathrm{V}}}{J^5} -$$

$$- \frac{27}{256}\frac{J_0'^3 J_0^{\mathrm{V}}}{J^6} - \frac{3}{784}\frac{J_0^{\mathrm{IV}2}}{J^4} + \frac{873}{7700}\frac{J_0' J_0''' J_0^{\mathrm{IV}}}{J^5} +$$

$$+ \frac{13149}{192500}\frac{J_0''^2 J_0^{\mathrm{IV}}}{J^5} - \frac{597}{175}\frac{J_0'^2 J_0'' J_0^{\mathrm{IV}}}{J^6} +$$

$$+ \frac{1215}{1792}\frac{J_0'^4 J_0^{\mathrm{IV}}}{J^7} + \frac{566}{6875}\frac{J_0'' J_0'''^2}{J^5} - \frac{15}{32}\frac{J_0'^2 J_0'''^2}{J^6} -$$

$$- \frac{18441}{13750}\frac{J_0' J_0''^2 J_0'''}{J^6} + \frac{81}{16}\frac{J_0'^3 J_0'' J_0'''}{J^7} - \frac{1701}{512}\frac{J_0'^5 J_0'''}{J^8} -$$

$$- \frac{10773}{68750}\frac{J_0''^4}{J^6} + \frac{729}{200}\frac{J_0'^2 J_0''^3}{J^7} - \frac{4212891}{352000}\frac{J_0'^4 J_0''^2}{J^8} +$$

$$+ \frac{1472571}{140800}\frac{J_0'^6 J_0''}{J^9} - \frac{16197}{5120}\frac{J_0'^8}{J^{10}}\Big\} a_0^2,$$

$$U_8 = \Big(\frac{2}{9J}\Big)^{1/3}\Big[\frac{a_1}{a_0^{3/2}}\Big(-\frac{11}{40}\frac{J_0'''}{J} + \frac{99}{80}\frac{J_0' J_0''}{J^2} -$$

$$- \frac{33}{32}\frac{J_0'^3}{J^3} + \frac{11}{10}JT\Big) - \frac{11}{32}\frac{a_1^2}{a_0^3}J + \frac{11}{4}\frac{a_2}{a_0^2}J +$$

$$+ T\Big(\frac{11}{560}\frac{J_0'''}{J} - \frac{641}{5600}\frac{J_0' J_0''}{J^2} + \frac{123}{1120}\frac{J_0'^3}{J^3}\Big) +$$

$$+ \frac{93}{400}JT^2 + \frac{7}{16}JT_S' - \frac{33}{16}J(k_1^2 + \delta_1^2) -$$

$$- \frac{7}{16}J(k_1 k_2 + \delta_1\delta_2) + \frac{7}{16}J(k_{1P}' + \delta_{1Q}') + \frac{1}{160}\frac{J_0^{\mathrm{VI}}}{J^2} -$$

$$- \frac{27}{320}\frac{J_0' J_0^{\mathrm{V}}}{J^3} - \frac{27}{140}\frac{J_0'' J_0^{\mathrm{IV}}}{J^3} - \frac{297}{448}\frac{J_0'^2 J_0^{\mathrm{IV}}}{J^4} -$$

$$- \frac{1}{8}\frac{J_0'''^2}{J^3} + \frac{99}{40}\frac{J_0' J_0'' J_0'''}{J^4} - \frac{231}{64}\frac{J_0'^3 J_0'''}{J^5} + \frac{297}{500}\frac{J_0''^3}{J^4} -$$

$$- \frac{6237}{800}\frac{J_0'^2 J_0''^2}{J^5} + \frac{35343}{2560}\frac{J_0'^4 J_0''}{J^6} - \frac{11781}{2048}\frac{J_0'^6}{J^7}\Big] a_0^{11/2},$$

$$U_9 = \Big\{\frac{a_1}{a_0^{3/2}}\Big(-\frac{9}{140}\frac{J_0'}{J}T + \frac{1}{28}\frac{J_0^{\mathrm{IV}}}{J^2} - \frac{1}{4}\frac{J_0' J_0'''}{J^3} -$$

$$- \frac{9}{50}\frac{J_0''^2}{J^3} + \frac{81}{80}\frac{J_0'^2 J_0''}{J^4} - \frac{81}{128}\frac{J_0'^4}{J^5}\Big) + \frac{1}{48}\frac{a_1^2}{a_0^3}\frac{J_0'}{J} -$$

$$- \frac{1}{3}\frac{a_2}{a_0^2}\frac{J_0'}{J} + T\Big(-\frac{11}{2100}\frac{J_0^{\mathrm{IV}}}{J^2} + \frac{22}{525}\frac{J_0' J_0'''}{J^3} +$$

$$+ \frac{131}{4200}\frac{J_0''^2}{J^3} - \frac{3149}{16800}\frac{J_0'^2 J_0''}{J^4} + \frac{279}{2240}\frac{J_0'^4}{J^5}\Big) +$$

$$+ \frac{J_0'}{J}\Big[-\frac{59}{4200}T^2 - \frac{1}{40}T_S' +$$

$$+ \frac{7}{15}(k_1^2 + \delta_1^2) + \frac{1}{40}(k_1 k_2 + \delta_1\delta_2) - \frac{1}{40}(k_{1P}' + \delta_{1Q}')\Big] -$$

$$- \frac{1}{2100}\frac{J_0^{\mathrm{VII}}}{J^3} + \frac{1}{120}\frac{J_0' J_0^{\mathrm{VI}}}{J^4} + \frac{9}{400}\frac{J_0'' J_0^{\mathrm{V}}}{J^4} -$$

$$- \frac{27}{320}\frac{J_0'^2 J_0^{\mathrm{V}}}{J^5} + \frac{1}{28}\frac{J_0''' J_0^{\mathrm{IV}}}{J^4} - \frac{27}{70}\frac{J_0' J_0'' J_0^{\mathrm{IV}}}{J^5} +$$

$$+ \frac{135}{224}\frac{J_0'^3 J_0^{\mathrm{IV}}}{J^6} - \frac{1}{4}\frac{J_0' J_0'''^2}{J^5} - \frac{333}{250}\frac{J_0''^2 J_0'''}{J^5} +$$

$$+ \frac{27}{8}\frac{J_0'^2 J_0'' J_0'''}{J^6} - \frac{405}{128}\frac{J_0'^4 J_0'''}{J^7} + \frac{81}{50}\frac{J_0' J_0''^3}{J^6} -$$

$$- \frac{729}{80}\frac{J_0'^3 J_0''^2}{J^7} + \frac{15309}{1280}\frac{J_0'^5 J_0''}{J^8} - \frac{2187}{512}\frac{J_0'^7}{J^9}\Big\} a_0^2,$$

$$U_{10} = \Big(\frac{2}{9J}\Big)^{1/3}\Big\{\frac{a_1}{a_0^{3/2}}\Big[T\Big(\frac{13}{1120}\frac{J_0''}{J} - \frac{13}{2240}\frac{J_0'^2}{J^2}\Big) -$$

$$- \frac{39}{2240}\frac{J_0^{\mathrm{V}}}{J^2} + \frac{39}{224}\frac{J_0' J_0^{\mathrm{IV}}}{J^3} + \frac{13}{40}\frac{J_0'' J_0'''}{J^3} -$$

$$- \frac{65}{64}\frac{J_0'^2 J_0'''}{J^4} - \frac{117}{80}\frac{J_0' J_0''^2}{J^4} + \frac{507}{128}\frac{J_0'^3 J_0''}{J^5} -$$

$$- \frac{507}{256}\frac{J_0'^5}{J^6}\Big] + \frac{a_2}{a_0^2}\Big(\frac{13}{40}\frac{J_0''}{J} - \frac{13}{32}\frac{J_0'^2}{J^2}\Big) +$$

$$+ T\Big(\frac{221}{61600}\frac{J_0^{\mathrm{V}}}{J^2} - \frac{8471}{215600}\frac{J_0' J_0^{\mathrm{IV}}}{J^3} - \frac{2333}{30800}\frac{J_0'' J_0'''}{J^3} +$$

$$+ \frac{173}{700}\frac{J_0'^2 J_0'''}{J^4} + \frac{10159}{28000}\frac{J_0' J_0''^2}{J^4} -$$

$$- \frac{11453}{11200}\frac{J_0'^3 J_0''}{J^5} + \frac{2379}{4480}\frac{J_0'^5}{J^6}\Big) +$$

$$+ \frac{J_0''}{J}\Big[\frac{19}{1750}T^2 + \frac{39}{4400}T_S' - \frac{1317}{3300}(k_1^2 + \delta_1^2) +$$

$$+ \frac{6}{275}(k_1 k_2 + \delta_1\delta_2) - \frac{6}{275}(k_{1P}' +$$

$$+ \delta_{1Q}')\Big] + \frac{J_0'^2}{J^2}\Big[-\frac{113}{8624}T^2 -$$

$$- \frac{13}{1760}T_S' + \frac{2081}{3520}(k_1^2 + \delta_1^2) - \frac{23}{1760}(k_1 k_2 + \delta_1\delta_2) +$$

$$+ \frac{23}{1760}(k_{1P}' + \delta_{1Q}')\Big] + \frac{13}{92400}\frac{J_0^{\mathrm{VIII}}}{J^3} - \frac{3}{800}\frac{J_0' J_0^{\mathrm{VII}}}{J^4} -$$

$$- \frac{1}{100}\frac{J_0'' J_0^{\mathrm{VI}}}{J^4} + \frac{13}{320}\frac{J_0'^2 J_0^{\mathrm{VI}}}{J^5} - \frac{3}{160}\frac{J_0''' J_0^{\mathrm{V}}}{J^4} +$$

$$+ \frac{351}{1600}\frac{J_0' J_0'' J_0^{\mathrm{V}}}{J^5} - \frac{819}{2240}\frac{J_0'^3 J_0^{\mathrm{V}}}{J^6} - \frac{9}{784}\frac{J_0^{\mathrm{IV}2}}{J^4} +$$

$$+ \frac{10821}{30800}\frac{J_0' J_0''' J_0^{\mathrm{IV}}}{J^5} + \frac{188379}{385000}\frac{J_0''^2 J_0^{\mathrm{IV}}}{J^5} -$$

$$- \frac{85539}{22400}\frac{J_0'^2 J_0'' J_0^{\mathrm{IV}}}{J^6} + \frac{2223}{896}\frac{J_0'^4 J_0^{\mathrm{IV}}}{J^7} +$$

$$+ \frac{44119}{55000}\frac{J_0'' J_0'''^2}{J^5} - \frac{13}{8}\frac{J_0'^2 J_0'''^2}{J^6} -$$

$$- \frac{43839}{6875}\frac{J_0' J_0''^2 J_0'''}{J^6} + \frac{741}{40}\frac{J_0'^3 J_0'' J_0'''}{J^7} - \frac{8151}{640}\frac{J_0'^5 J_0'''}{J^8} -$$

$$- \frac{96903}{13750}\frac{J_0''^4}{J^6} + \frac{13338}{1000}\frac{J_0'^2 J_0''^3}{J^7} - \frac{4209219}{176000}\frac{J_0'^4 J_0''^2}{J^8} +$$

$$+ \frac{13237731}{281600}\frac{J_0'^6 J_0''}{J^9} - \frac{2409207}{163840}\frac{J_0'^8}{J^{10}}\Big\} a_0^{13/2}. \quad (5.1)$$

We shall now assume that the emitter is either the plane $x = 0$, or the cylinder $R = R_0$, or the sphere $r = r_0$. In the first two cases the curvatures of the surfaces orthogonal to the emitter are zero, while in the last case $k_2 \neq 0$ but it enters into the formulas only through the Gaussian curvature $k_1 k_2$, and therefore also cancels out. Let the emission-current density J be

$$J = 3 + \sin t. \qquad (5.2)$$

Equations (2.5) and (5.1) enable us to determine the potential and the velocity. To calculate the current density at any point, we shall use the relations obtained by integrating the equation describing the conservation of current and the fact that the displacement current at the emitter is zero:

$$J(t, x) = J(t) - \frac{\partial^2 \varphi}{\partial t\, \partial x}, \qquad J(t, R) = \frac{J(t)}{R} - \frac{\partial^2 \varphi}{\partial t\, \partial R}$$

$$J(t, r) = \frac{J(t)}{r^2} - \frac{\partial^2 \varphi}{\partial t\, \partial r}.$$

The characteristic linear dimension a in the plane case can then be taken to be the distance between the electrodes, whereas in the cylindrical and spherical cases it can be taken to be the radius of the emitter. The results can be conveniently expressed in terms of the dimensionless variables related to the Child-Langmuir solution for the planar diode [4]. Transformation from the variables of §1 (φ°) to these new dimensionless variables (φ°_L), subject to Eq. (5.2), can be achieved with the aid of the factors

$$\varphi_L{}^\circ = 2\,(^2/_{27})^{1/3}\,\varphi^\circ,$$

$$v_L{}^\circ = (^2/_{27})^{1/6}\,v^\circ, \qquad J_L{}^\circ = {}^1/_3\, J^\circ.$$

For a planar diode $g_{11} = 1$, i.e., $a_0 = 1$, $a_k = 0$ $(k > 0)$, $T = 0$. For cylindrical and spherical diodes we can take x^1 to be the logarithm of the radius ($\ln R$, $\ln r$) and, consequently, $g_1 = \exp(2x^1)$, $a_0 = 1$, $a_1 = 2$, $a_2 = 2$, and so on. Moreover, $T = -1/R$, $T'_S = 1/R^2$ in the cylindrical case, and $T = -2/r$, $T'_S = 2/r^2$ in the spherical case. Figure 1 shows plots of $\varphi^\circ_L(t,1) = \varphi_2$, $v^\circ_L(t,1) = v_2$, $J^\circ_L(t,1) = J_2$ and $J^\circ_L(t,0) = 1 + (\sin t)/3$. The subscript 2 refers to quantities evaluated at the collector. Figure 2 shows the analogous curves for the cylindrical and spherical diodes when the distance between the electrodes is equal to the radius of the emitter.

The above solutions describe the high-frequency conditions in planar, cylindrical, and spherical diodes for space-charge-limited emission. The frequency $\omega^\circ = 1$ corresponds to the dimensional frequency $\omega \sim U/a$, where a is a characteristic linear dimension, and U is the velocity acquired by a particle at the collector of the planar diode $0 \leq x \leq a$, in which the current density is J. As the frequency and amplitude of the high-frequency component are reduced, the convergence of the series improves rapidly, especially in the case of φ.

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